criminator can be made with a circular waveguide detector, where TE_{01} and TE_{11} modes interfere. The detectors must be situated in the same transverse plane but at opposite walls in such a way that the coupling with the TE_{11} mode is maximum. A small asymmetry is already sufficient to excite the TE_{01} mode. Usually the guide is below cutoff for this mode, which means that it forms a resonator for the TE_{01} mode. As the detector currents due to this mode are antiphase and those caused by the TE_{11} mode are in phase, the differential voltage of the two detectors as a function of frequency gives a discriminator curve.

If condition (1) is fulfilled, wall-current detectors with identical frequency characteristics can be constructed for frequencies lower and higher than X-band frequencies and for other types of transmission lines.

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First-Order Theory for Oblate and Prolate Anisotropic Artificial Dielectrics*

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Summary—**Equivalent expressions for the electric permittivity and magnetic permeability tensors of artificial dielectrics are derived. These are expressed as functions of particle dimension, shape and density and also as a function of the incident electromagnetic beam direction with respect to the orientation of the particle. Only the case of a uniform density of equally oriented particles is considered. The results are valid in first order for prolate and oblate spheroids. Spheres and disks are obtained as limiting cases.**

INTRODUCTION

THE DEVELOPMENT of microwave applications in the two last decades caused an extension of optical techniques to this region of the electromagnetic spectrum. In 1948, Kock[1] proposed that a three dimensional lattice of identical metallic particles, disposed as atoms in a crystal, would act as an "artificial" dielectric. Equivalent arrangements may be obtained in different ways,[2] but this paper will be restricted to the important case of lattices of oriented particles.

Some of the more important applications of artificial dielectrics are: phase delay lenses, filters and polarization transformers. The physical mechanism is very simple: as a convenient wavelength electromagnetic wave is incident on a metallic particle, electric dipole moments and currents are induced in the particle. This will produce a phase delay in the electric and magnetic vector fields, in the same way in which it occurs in natural dielectrics. These delays depend upon the magnitude of the induced moments and currents, which in turn depend on the shape and the orientation of particles. These induced moments and currents can be accounted for by the proper permittivity and permeability tensors.

THE DEBYE AND MOSSOTI APPROXIMATIONS

If $\boldsymbol{E}^1$ is the local field, $\boldsymbol{p}$ the mean dipole moment per particle, then, under a quasi-static field approximation, $\boldsymbol{p}=\alpha\boldsymbol{E}^1$, where α is the polarizability. The polarization vector $\boldsymbol{P}$ becomes $\boldsymbol{P}=N\boldsymbol{p}=N\alpha\boldsymbol{E}^1$, where N is the number of dipoles per unit volume. As $\boldsymbol{P}=\boldsymbol{D}-\epsilon_0\boldsymbol{E}=(\epsilon-\epsilon_0)\boldsymbol{E}$, where $\boldsymbol{E}$ is the applied field, there is

$$N\alpha\boldsymbol{E}^1 = (\epsilon - \epsilon_o)\boldsymbol{E}.$$

Debye's approximation consists of neglecting the contribution of other particles to $\boldsymbol{E}^1$ and in the consequent identification of $\boldsymbol{E}$ with $\boldsymbol{E}^1$. This leads to

$$\epsilon_r = \frac{N\alpha}{\epsilon_o} + 1,$$

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[1] W. E. Kock, "Metal-lens antennas," PROC. IRE, vol. 34, pp. 828–836; November, 1946.

[2] John Brown, "Artifical dielectrics" in "Progress in Dielectrics," J. B. Birks and J. H. Schulman, Ed., John Wiley and Sons, Inc., New York, N. Y., vol. 2, pp. 193–225; 1960.

where

$$\epsilon_r = \frac{\epsilon}{\epsilon_o}.$$

A similar approximation gives:

$$\mu_r = \mu_o\left[1 + \frac{M}{H}\right].$$

In order to avoid resonance effects the particle dimensions must be less than $\lambda/4$,[3] and in order to avoid diffraction λ must be longer than the distance between neighboring particles.

Mossoti's approximation[4] takes into consideration the contribution of the polarization of distant particles to the local field acting on a reference particle and neglects the contribution of near particles. This is a better approximation for natural dielectrics, where packing is so high that one cannot neglect mutual interaction of dipole moments and induced currents.

Nevertheless, Brown[2] has shown that for a cubic lattice of spheres Debye's and Mossoti's approximations are equivalent, if the sphere diameter is $\leq 0.7d$, where d is the distance between neighboring particles. So if it is assumed that this is also true for spheroids, the simpler Debye theory can be used to avoid the mathematical complications arising from Mossoti's approach.

It is then assumed that the longest dimension of the spheroid is smaller than 0.7 the lattice constant.

The Prolate Case[5]

Here the prolate coordinates[6] which can be related to the cartesian system by

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad z = c\eta\xi$$

and

$$\rho = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \tag{2}$$

where $\infty > \xi \geq 1$, $-1 < \eta < 1$ and $2c$ is the focal distance, will be used. The relation between the prolate system and a spherical reference system is given by

$$\Phi = \Phi, \qquad r = c\sqrt{\xi^2 + \eta^2 - 1}, \qquad \cos\theta = \frac{\eta\xi}{\sqrt{\xi^2 + \eta^2 - 1}}$$

$$\sin\theta = \frac{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}{\sqrt{\xi^2 + \eta^2 - 1}} = \frac{\rho}{r} \tag{3}$$

and, for $\xi \gg 1$, (3) reduces to

$$r = c\xi, \qquad \cos\theta = \eta, \qquad \Phi = \Phi. \tag{4}$$

For oblique incidence, the electric and its associated magnetic field may be described by:

$$\begin{aligned} E_{(x)}{}^i &= E_{ox}\hat{x}e^{ikz} & H_{(y)}{}^i &= H_{oy}\hat{y}e^{ikz} \\ E_{(y)}{}^i &= E_{oy}\hat{y}e^{ikx} & H_{(z)}{}^i &= H_{oz}\hat{z}e^{ikx} \\ E_{(z)}{}^i &= E_{oz}\hat{z}e^{iky} & H_{(x)}{}^i &= H_{ox}\hat{x}e^{iky}. \end{aligned} \tag{5}$$

$\hat{x}(\hat{y}, \hat{z})$ is a unitary vector in the $x(y, z)$ direction,

$$H_o = \left(\frac{\epsilon_o}{\mu_o}\right)^{1/2} E_o,$$

and a harmonic time dependent factor of the form $e^{-i\omega t}$ is implicitly understood.

As done by Rayleigh,[7] the incident field in the neighborhood of the spheroid may be approximated by its first term in a series development:

$$E_{1x}{}^i = E_{ox}\hat{x}, \quad E_{1y}{}^i = E_{oy}\hat{y} \text{ and } E_{1z}{}^i = E_{oz}\hat{z}. \tag{6}$$

The subscript "1" means a first-order approximation.

The potential function related to $\boldsymbol{E}$ must satisfy Laplace's equation which in our case can be solved by separation of variables. This method leads to Legendre equations. The characteristic potential functions in our reference system become

$$\psi_{\substack{e\\o}mn}{}^{(1)} = (-1)^m P_n{}^m(\eta) P_n{}^m(\xi) \begin{matrix}\cos\\ \sin\end{matrix}\, m\Phi$$

$$\psi_{\substack{e\\o}mn}{}^{(2)} = (-1)^m P_n{}^m(\eta) Q_n{}^m(\xi) \begin{matrix}\cos\\ \sin\end{matrix}\, m\Phi, \tag{7}$$

where $P_n{}^m$ and $Q_n{}^m$ are associated Lengendre functions of first and second kind respectively.[6]

Being restricted to quasi-static fields, the conditions

$$\nabla \times \boldsymbol{A} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{A} = 0 \tag{8}$$

must be fulfilled by $\boldsymbol{E}_1$ and $\boldsymbol{H}_1$.

[3] J. D. Kraus, "Antennas," McGraw-Hill Book Company, Inc., New York, N. Y. p. 393; 1950.

[4] See for instance, A. von Hippel, "Dielectrics and Waves," John Wiley and Sons, Inc., New York, N. Y., pp. 178–181; 1954.

[5] Scattering by a prolate spheroid with the incident wave propagated in parallel to the axis of symmetry has been considered by Schultz. (F. V. Schultz, "Scattering by a Prolate Spheroid," University of Michigan, Ann Arbor; Rept. U.M.M. 42; 1950.) A very through treatment of electromagnetic scattering by ellipsoids in the third approximation was given by A. F. Stevenson. His approach is somewhat more basic than the authors' and the reader particularly interested in scattering is referred to the following articles: A. F. Stevenson, "Solution of electromagnetic scattering problem as power series in the ratio (dimension of scatter)/wavelength," *J. Appl. Phys.* vol. 24, pp. 1134–1142, 1953; and "Electromagnetic scattering by an ellipsoid in the third approximation," *J. Appl. Phys.*, vol. 24, pp. 1143–1151, 1953.

[6] W. R. Smythe, "Static and Dynamic Electricity," McGraw-Hill Book Company, Inc., New York, N. Y., pp. 158–159; 1939.

[7] J. W. S. Rayleigh, "On the incidence of aerial and electric waves upon small obstacles in the form of ellipsoids or elliptic cylinders, and on the passage of electric waves through a circular aperture in a conducting screen," *Phil Mag.*, vol. 44, pp. 23–52; 1897. Rayleigh only considered scattering, where the direction of propagation was along one principal axis of the spheroid with the electric vector along another principal axis.

A typical solution for (8) is $\boldsymbol{A}=\nabla\psi$ where ψ is given by (7). The notation, introduced by C. T. Tai[8] and based on the set of solutions of (7), *i.e.*,

$$S_{\substack{e\\o}mn} = c\nabla\psi_{\substack{e\\o}mn}, \tag{9}$$

will be adopted here. Direct examination of $S_{\substack{e\\o}mn}$ and comparison with (2) shows that

$$\hat{x} = S_{e11}^{(1)} = c\nabla\psi_{e11}^{(1)} = c\nabla[P_1^1(\xi)P_1^1(\eta)\cos\theta]$$
$$\hat{y} = S_{o11}^{(1)} = c\nabla\psi_{o11}^{(1)} \qquad \hat{z} = S_{e01}^{(1)} = c\nabla\psi_{e01}^{(1)}. \tag{10}$$

The scattered field ($\boldsymbol{E}^s$, $\boldsymbol{H}^s$) will now be constructed. The near field is a quasi-static field, but the distant field behaves as a spherical wave. Rayleigh's method consists of identifying the near field with a field possessing the characteristics of a spherical wave in a region intermediate to the distant and near field regions.

On the surface of the reference spheroid the following conditions must be fulfilled:

$$\hat{\xi}_o \times (\boldsymbol{E}^i + \boldsymbol{E}^s) = 0, \qquad \hat{\xi}_o\cdot(\boldsymbol{H}^i + \boldsymbol{H}^s) = 0, \tag{11}$$

where $\hat{\xi}_0$ is the unitary vector perpendicular to the spheroid surface.

As the scattered field must be divergent, there is, in the vicinity of the spheroid,

$$E_{1x}{}^s = \alpha_x E_o S_{e11}{}^{(2)} \qquad H_{1x}{}^s = \beta_x H_o S_{o11}{}^{(2)}$$
$$E_{1y}{}^s = \alpha_y E_o S_{o11}{}^{(2)} \qquad H_{1y}{}^s = \beta_y H_o S_{e11}{}^{(2)}$$
$$E_{1z}{}^s = \alpha_z E_o S_{o01}{}^{(2)} \qquad H_{1z}{}^s = \beta_z H_o S_{e01}{}^{(2)}. \tag{12}$$

Now, by combining (12), (11), (6), (5) and (10),

$$\alpha_x = \alpha_y = -\frac{P_1^1(\xi_o)}{Q_1^1(\xi_o)},$$
$$\alpha_z = \frac{P_1(\xi_o)}{Q_1(\xi_o)}$$
$$\beta_x = \beta_y = -\left[\frac{\frac{\partial}{\partial\xi}P_1^1(\xi)}{\frac{\partial}{\partial\xi}Q_1^1(\xi)}\right]_{\xi=\xi_0}$$
$$\beta_z = -\left[\frac{\frac{\partial}{\partial\xi}P_1(\xi)}{\frac{\partial}{\partial\xi}Q_1(\xi)}\right]_{\xi=\xi_0} \tag{13}$$

are obtained.

Eqs. (12) and (13) describe in first order the scattered near field. Proceeding with Rayleigh's method, from (10),

$$\begin{aligned}
\begin{matrix}\hat{x}\\ \hat{y}\end{matrix} = S_{\substack{e\\o}11}{}^{(1)} &= \left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2}\frac{\partial P_1^1(\eta)}{\partial\eta}P_1^1(\eta)\begin{matrix}\cos\\ \sin\end{matrix}\Phi\hat{\eta}\\
&+\left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2}\frac{\partial P_1^1(\xi)}{\partial\xi}P_1^1(\eta)\begin{matrix}\cos\\ \sin\end{matrix}\Phi\hat{\xi}\\
&\mp\frac{1}{\sqrt{(\xi^2-1)(1-\eta^2)}}P_1^1(\eta)P_1^1(\xi)\begin{matrix}\sin\\ \cos\end{matrix}\Phi\hat{\Phi}\\
\hat{z} = S_{e01}{}^{(1)} &= \left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2}\frac{\partial P_1(\eta)}{\partial\eta}P_1(\xi)\hat{\eta}\\
&+\left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2}P_1(\eta)\frac{\partial P_1(\xi)}{\partial\xi}\hat{\xi}.
\end{aligned} \tag{14}$$

For $\xi\gg 1$, Legendre associated functions and derivatives become

$$Q_1^1(\xi) \doteq -\frac{2}{3\xi^2}, \qquad \frac{\partial}{\partial\xi}Q_1^1 \doteq \frac{4}{3\xi^3}, \qquad Q_1(\xi) \doteq \frac{1}{3\xi^2},$$
$$\frac{\partial Q_1}{\partial\xi} \doteq -\frac{2}{3\xi^3}, \qquad P_1^1(\xi) = \xi, \qquad \frac{\partial}{\partial\xi}P_1^1 \doteq 1,$$
$$P_1(\xi) \doteq \xi, \qquad \frac{\partial}{\partial\xi}P_1 \doteq 1, \qquad P_1(\eta) \doteq \cos\theta,$$
$$P_1^1(\eta) \doteq \sin\theta, \qquad \frac{\partial}{\partial\eta}P_1 \doteq 1, \qquad \frac{\partial}{\partial\eta}P_1^1 \doteq -\operatorname{cotg}\theta,$$

so that the scattered near field is given by

$$\begin{aligned}
E_{1x}{}^s &= \alpha_x E_{ox}\left(\frac{c}{r}\right)^3\frac{2}{3}\\
&\cdot[2\sin\theta\cos\Phi\hat{r} - \cos\theta\cos\Phi\hat{\theta} + \sin\Phi\hat{\Phi}]\\
E_{1z}{}^s &= \alpha_y E_{oy}\left(\frac{c}{r}\right)^3\frac{2}{3}\\
&\cdot[2\sin\theta\sin\Phi\hat{r} - \cos\theta\sin\Phi\hat{\theta} - \sin\Phi\hat{\Phi}]\\
E_{1z}{}^s &= \alpha_z E_{oz}\left(\frac{c}{r}\right)^3\frac{2}{3}[-2\cos\theta\hat{r} + \sin\theta\hat{\theta}].
\end{aligned} \tag{15}$$

According to Stratton,[9] there are two sets of spherical wave functions which can be used to represent the scattered field,

$$\begin{aligned}
\boldsymbol{N}_{\substack{e\\o}mn} &= \frac{(n+1)n}{Kr}Z_n(Kr)P_n{}^m(\cos\theta)\begin{matrix}\sin\\ \cos\end{matrix}m\Phi\hat{r}\\
&+\frac{i}{Kr}\frac{\partial}{\partial r}[rZ_n(Kr)]\frac{\partial}{\partial\theta}P_n{}^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}m\Phi\hat{\theta}\\
&\mp\frac{m}{Kr\sin\theta}\frac{\partial}{\partial r}[rZ_n(Kr)]P_n{}^m(\cos\theta)\begin{matrix}\sin\\ \cos\end{matrix}m\Phi\hat{\Phi}
\end{aligned} \tag{16}$$

[8] C. T. Tai, "Quasi-Static Solution for Diffraction of a Plane Electromagnetic Wave by a Small Oblate Spheroid," Stanford Res. Inst., Menlo Park, Calif.; Tech. Rept. No. 24; 1952.

[9] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Company, Inc. New York, N. Y., pp. 414–416; 1941.

and

$$M_{\substack{e\\o}mn} = \mp \frac{m}{\sin\theta} Z_n(Kr) P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix}\, m\Phi\hat{\theta} - Z_n(Kr) \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\Phi\hat{\Phi},$$

where $Z_n(Kr)$ is a spherical Bessel function of order n. Now, for $r \ll \lambda$,

$$Z_1(Kr) \doteq \frac{i}{K^2r^2} \quad \text{and} \quad \frac{\partial}{\partial r}[rZ_1(Kr)] = \frac{i}{K^2r^2}$$

and $N_{\substack{e\\o}11}$ and N_{e01} become

$$N_{\substack{e\\o}11} \doteq -\frac{i}{K^3r^3} \cdot \left[2\sin\theta \begin{matrix}\cos\\ \sin\end{matrix}\, \Phi\hat{r} - \cos\theta \begin{matrix}\cos\\ \sin\end{matrix}\, \Phi\hat{\theta} \pm \sin\Phi\hat{\Phi}\right]$$

$$N_{e01} \doteq \frac{i}{K^3r^3}[-2\cos\theta\hat{r} + \sin\theta\hat{\theta}]. \tag{17}$$

By matching (15) with (17) in the intermediate region,

$$\begin{aligned} E_{1x}^s &= \gamma_x N_{e11} & \gamma_x &= i\frac{2}{3}K^3c^3\alpha_x E_{ox} \\ E_{1y}^s &= \gamma_y N_{o11} \quad \text{where} & \gamma_y &= i\frac{2}{3}K^3c^3\alpha_y E_{oy} \\ E_{1z}^s &= \gamma_z N_{e01} & \gamma_z &= -\frac{i}{3}K^3c^3\alpha_z E_{oz} \end{aligned} \tag{18}$$

which gives the scattered near field. In order to obtain the distant field (E^w, H^w),

$$Z_1(Kr) \doteq -\frac{e^{iKr}}{Kr}$$

is put for $r \gg \lambda$ and for the distant scattered field the following is obtained:

$$\begin{aligned} E_{1x}^W &= \frac{2}{3}K^2c^3\alpha_x[\cos\theta\cos\Phi\hat{\theta} - \sin\Phi\hat{\Phi}]\frac{e^{iKr}}{r}E_{ox} \\ E_{1y}^W &= \frac{2}{3}K^2c^3\alpha_y[\cos\theta\sin\Phi\hat{\theta} + \cos\Phi\hat{\Phi}]\frac{e^{iKr}}{r}E_{oy} \\ E_{1z}^W &= \frac{1}{3}K^2c^3\alpha_z[\sin\theta\hat{\theta}]\frac{e^{iKr}}{r}E_{oz}. \end{aligned} \tag{19a}$$

By the same method,

$$\begin{aligned} H_{1x}^W &= \frac{2i}{3}K^2c^3\beta_x[\cos\theta\sin\Phi\hat{\theta} + \cos\hat{\Phi}]\frac{e^{iKr}}{r}H_{ox} \\ H_{1y}^W &= \frac{2i}{3}K^2c^3\beta_y[\cos\theta\cos\Phi\hat{\theta} + \sin\Phi\hat{\Phi}]\frac{e^{iKr}}{r}H_{oy} \\ H_{1z}^W &= \frac{i}{3}K^2c^3\beta_z[\sin\theta\hat{\theta}]\frac{e^{iKr}}{r}H_{oz}. \end{aligned} \tag{19b}$$

Debye's method consists of determining the distant field due to an equivalent dipole on the particles and of identifying this field with the field previously obtained. Now, expressing the field due to a dipole of components $p_{x_i}(x_1 = x,\ x_2 = y,\ x^3 = z)$,

$$E_{x_i}^W = \frac{p_{x_i}}{4\pi E_o}\nabla\times\nabla\times\left(\hat{x}_i\frac{e^{iKr}}{r}\right)$$

which for $r \gg \lambda$ becomes

$$\begin{aligned} E_x^W &\doteq \frac{p_xK^2}{4\pi\epsilon_o}[\cos\theta\cos\Phi\hat{\theta} - \sin\Phi\hat{\Phi}]\frac{e^{iKr}}{r} \\ E_y^W &\doteq \frac{p_yK^2}{4\pi\epsilon_o}[\cos\theta\sin\Phi\hat{\theta} + \cos\Phi\hat{\Phi}]\frac{e^{iKr}}{r} \\ E_z^W &\doteq \frac{p_zK^2}{4\pi\epsilon_o}[-\sin\theta\hat{\theta}]\frac{e^{iKr}}{r}\cdot \end{aligned} \tag{20}$$

By identifying (19) with (20),

$$p_{\substack{x\\y}} = \frac{8\pi E_o}{3}c^3E_{o\substack{x\\y}}\alpha_{\substack{x\\y}} \quad \text{and} \quad p_z = -\frac{4\pi E_o}{3}c^3E_{oz}\alpha_z. \tag{21a}$$

By an analogous procedure,

$$m_{\substack{x\\y}} = \frac{8\pi}{3}c^3\beta_{\substack{x\\y}}H_{o\substack{x\\y}} \quad \text{and} \quad m_z = -\frac{4\pi E_o}{3}c^3H_{oz}\beta_z. \tag{21b}$$

In an anisotropic medium, the relationships between D and E and between B and H are given by the second order symmetric tensors $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$. These relationships will be determined without an explicit calculation of the components of $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$.

Take E and H in the y and z directions, respectively. θ_1 is a rotation around the y axis and θ_2 is a rotation around the x axis after θ_1. This may be represented by a rotation matrix R, such that $RX = X'$, where

$$R = \begin{vmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ \sin\theta_1\sin\theta_2 & \cos\theta_2 & \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2 & -\sin\theta_2 & \cos\theta_1\cos\theta_2 \end{vmatrix},$$

X' = reference system of the spheroid,
X = reference system defined by the incident field,

so that

$$E_o\hat{z} = E_o(-\sin\theta_1\hat{x}' + \cos\theta_1\sin\theta_2\hat{y}' + \cos\theta_1\cos\theta_2\hat{z}')$$
$$H_o\hat{y} = H_o(\cos\theta_2\hat{y}' - \sin\theta_2\hat{z}')$$

and, hence from (21a) and (21b),

$$p_x' = \frac{8\pi E_o}{3}c^3\alpha_x(-\sin\theta_1)E_o$$

$$p_y' = \frac{8\pi E_o}{3}c^3\alpha_y(\cos\theta_1\sin\theta_2)E_o$$

$$p_z' = -\frac{8\pi E_o}{3} c^3\alpha_z(\cos\theta_1\cos\theta_2)E_o$$

$$m_x' = 0 \qquad m_{y,z}' = \pm\frac{8\pi c^3}{3}\beta_{y,z}\,{}^{\cos}_{\sin}\,\theta_2 H_o$$

as

$$E_z = E_o + \frac{P_z}{E_z} \quad\text{and}\quad P_z = AP_x' + BP_y' + CP_z'.$$

Where A, B, C are the direction cosines,

$$\epsilon_z = \epsilon_o\left\{1 + \frac{4\pi c^3}{3}N[\alpha_t(\sin^2\theta_1 + \cos^2\theta_1\sin^2\theta_2) + \alpha_e(\cos^2\theta_1\cos^2\theta_2)]\right\}$$

$$\mu_y = \mu_o\left[1 - \frac{4\pi c^3}{3}N(\beta_t\cos^2\theta_2 + \beta_e\sin^2\theta_2)\right], \tag{22}$$

where

$$\alpha_t = 2\alpha_x = 2\alpha_y = -2\frac{P_1^1(\xi_o)}{Q_1^1(\xi_o)} = -\frac{2}{\tanh^{-1}\frac{1}{\xi_o} - \frac{\xi_o}{\xi_o^2 - 1}}$$

$$\alpha_e = -\alpha_z = \frac{P_1(\xi_o)}{P_1(\xi_o)} = \frac{1}{\tanh^{-1}\frac{1}{\xi_o} - \frac{1}{\xi_o}}$$

$$\beta_t = -2\beta_x = -2\beta_y = 2\left[\frac{\frac{\partial}{\partial\xi}P_1^1(\xi)}{\frac{\partial}{\partial\xi}Q_1^1(\xi)}\right]_{\xi=\xi_0}$$

$$= \frac{1}{\tanh^{-1}\frac{1}{\xi_o} - \frac{\xi_o}{\xi_o^2 - 1}}$$

$$\beta_e = \beta_z = -\left[\frac{\frac{\partial}{\partial\xi}P_1(\xi)}{\frac{\partial}{\partial\xi}Q_1(\xi)}\right] = -\frac{2}{\tanh^{-1}\frac{1}{\xi_o} + \frac{2-\xi_o^2}{\xi_o^3 - \xi_o}}.$$

By putting $\boldsymbol{E} = E_0\hat{y}$, $\boldsymbol{H} = H_0\hat{z}$ and assuming propagation along x,

$$\epsilon_y = \epsilon_o\left[1 + \frac{4\pi c^3}{3}N(\alpha_t\cos^2\theta_2 + \alpha_e\sin^2\theta_2)\right]$$

$$\mu_z = \mu_o\left\{1 - \frac{4\pi c^3}{3}N[\beta_t(\sin^2\theta_1 + \cos^2\theta_1\sin^2\theta_2) + \beta_e\cos^2\theta_1\cos^2\theta_2]\right\}. \tag{23}$$

Eqs. (22) and (23) are equivalent to the knowledge of tensors $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ and allow the calculation of the optical properties of this medium.

For $\xi_0 \to \infty$, prolate spheroids become spheres and the medium becomes isotropic. Eqs. (22) and (23) give the sphere case

$$[\beta_t]_{\xi_0\to\infty} = [\beta_e]_{\xi_0\to\infty} = \frac{3r^3}{2c^3}$$

$$[\alpha_t]_{\xi_0\to\infty} = [\alpha_e]_{\xi_0\to\infty} = \frac{3r^3}{c^3}$$

and, consequently,

$$\epsilon_z = \epsilon_y = \epsilon_o(1 + 4\pi Nr^3)$$

$$\mu_z = \mu_y = \mu_o(1 - 2\pi Nr^3)$$

which are the well known expressions for the case of a sphere as found in the literature.[3]

The Oblate Case

The method is identical to that of prolate case, so only the differences in the main steps will be pointed out.

In this case, the reference system where Laplace equation can be solved by separation of variables is an oblate system as described by Smythe.[6] The potential functions obtained are

$$\psi_{{}^e_o mn}^{(1)} = (i)^{-n}P_n^m(\eta)P_n^m(i\xi)\,{}^{\cos}_{\sin}\,m\Phi$$

$$\psi_{{}^e_o mn}^{(2)} = (i)^{-n}P_n^m(\eta)Q_n^m(i\xi)\,{}^{\cos}_{\sin}\,m\Phi.$$

The scattered near field is given by

$$E_{1x}^s = i\alpha_x E_{ox}S_{e11}^{(2)} \qquad H_{1x}^s = i\beta_x H_{ox}S_{o11}^{(2)}$$

$$E_{1y}^2 = i\alpha_y E_{oy}S_{o11}^{(2)} \qquad H_{1y}^s = i\beta_y H_{oy}S_{e11}^{(2)}$$

$$E_{1z}^s = i\alpha_z E_{oz}S_{e01}^{(2)} \qquad H_{1z}^s = i\beta_z H_{oz}S_{e01}^{(2)}$$

$$\alpha_x = \alpha_y = -\frac{P_1^1(i\xi_o)}{Q_1^1(i\xi_o)}, \qquad \alpha_z = -\frac{P_1(i\xi_o)}{Q_1(i\xi_o)}$$

$$\beta_x = \beta_y = -\left[\frac{\frac{\partial}{\partial\xi}P_1^1(i\xi)}{\frac{\partial}{\partial\xi}Q_1^1(i\xi)}\right]_{\xi=\xi_o},$$

$$\beta_z = -\left[\frac{\frac{\partial}{\partial\xi}P_1(i\xi)}{\frac{\partial}{\partial\xi}Q_1(i\xi)}\right]_{\xi=\xi_o}.$$

The distant field in this case has the same expression (19) as in the prolate case except for the α and β factors. By following the same method, ϵ_z, μ_y, ϵ_y, μ_z as given by

(22) and (23) is obtained, but here α_t, α_e, β_t and β_e are given by

$$\alpha_t = -2i\alpha_x = -2i\alpha_y = \frac{2}{\tan^{-1}\frac{1}{\xi_o} - \frac{\xi_o}{\xi_o^2+1}},$$

$$\alpha_e = i\alpha_z = -\frac{1}{\tan^{-1}\frac{1}{\xi_o} - \frac{1}{\xi_o}}$$

$$\beta_t = 2i\beta_x = 2i\beta_y = -\frac{2}{\tan^{-1}\frac{1}{\xi_o} - \frac{\xi_o^2+2}{\xi_o^3+\xi_o}},$$

$$\beta_e = -i\beta_z = \frac{1}{\tan^{-1}\frac{1}{\xi_o} - \frac{\xi_o}{\xi_o^2+1}}. \tag{24}$$

Eqs. (22)–(24) describe the optical properties of the medium for the oblate case.

For $\xi_0 \to 0$, the spheroid becomes a disk and according to (22)–(24),

$$[\alpha_t]_{\xi_0=0} = 2[\beta_e]_{\xi_0=0} = \frac{4}{\pi}, \qquad \alpha_e = \beta_t = 0.$$

Then from (22),

$$\epsilon_z = \epsilon_o\left[1 + \frac{16}{3}c^3N(\sin^2\theta_1 + \cos^2\theta_1\sin^2\theta_2)\right]$$

$$\mu_y = \mu_o\left(1 - \frac{8}{3}c^3N\sin^2\theta_2\right)$$

which is identical to the expressions given in the literature (see for example Kraus[3]) for disks, if it is noticed that $2c$ becomes the disk diameter.

Numerical Calculations

Fig. 1 and Fig. 2 show δ_1/N and δ_2/N as a function of ξ_0 for the prolate and oblate cases, respectively. δ_1 and δ_2 are the incremental electric permittivity and magnetic permeability defined by

$$\delta_1 = \epsilon_r - 1 \quad \text{where} \quad \epsilon_r = \frac{\epsilon}{\epsilon_o}$$

$$\delta_2 = 1 - \mu_r \quad \text{where} \quad \mu_r = \frac{\mu}{\mu_o}.$$

In Figs. 1 and 2, the upper group of curves refers to δ_1 and the lower group to δ_2. Several incidence angles are considered. $\boldsymbol{E}$ and $\boldsymbol{H}$ are in the $\hat{z}$ and $\hat{y}$ directions, respectively, and the propagation along the $\hat{x}$ direction.

The largest dimension of the spheroids is fixed as 2 cm by making $c\xi_0 = 1.0$ cm and N is the number of particles per cubic meter.

By means of Figs. 1 and 2, one may determine the optical properties of the medium in a first-order approximation for any desired shape of spheroids.

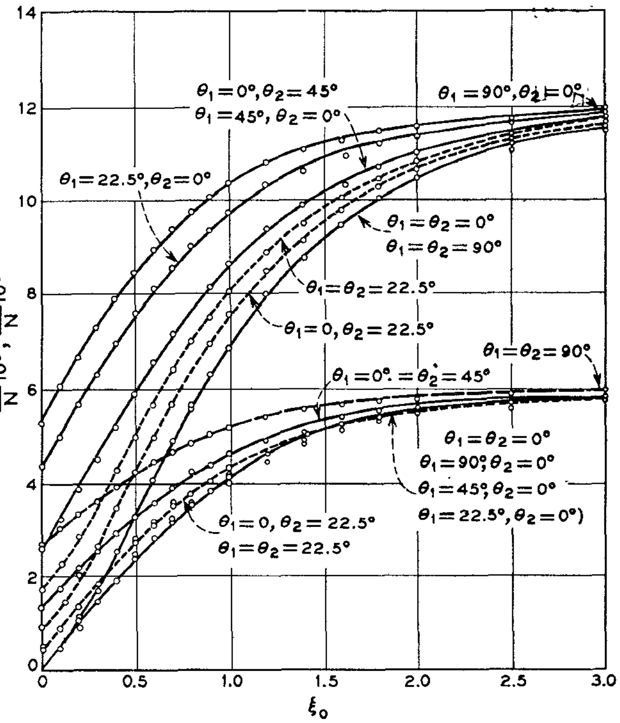

Fig. 1—Plots of the incremental electric permitivity and magnetic permeability as functions of the eccentricity ξ_o for the prolate case. The upper group of curves refers to δ_1 and the lower group to δ_2.

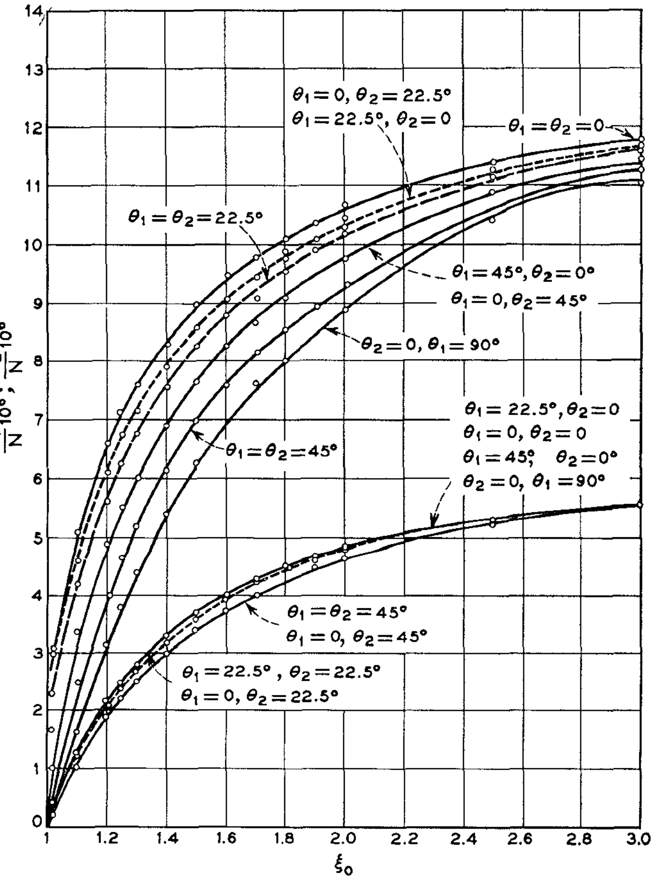

Fig. 2—The oblate case. The incremental electric permittivity and magnetic permeability as functions of the eccentricity. The upper group of curves refers to δ_1 and the lower group to δ_2.